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Minimal Models with Integrable Local Defects

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We describe a general way of constructing integrable defect theories as perturbations of conformal field theory by local defect operators. The method relies on folding the system onto a boundary field theory of *twice* the central charge. The classification of integrable defect theories obtained in this way parallels that of integrable bulk theories which are a perturbation of the tensor product of *two* conformal field theories. These include local defect perturbations of all $c < 1$ minimal models, as well as of the coset theories based on $SO(2n)$, obtained in this way. We discuss in detail the former case of all the Virasoro minimal models. In the Ising case our construction corresponds to having a spin field as a defect operator; in the folded formulation this is mapped onto an orbifolding of the boundary sine-Gordon theory at $\beta^2/8\pi = 1/8$, or a version of the anisotropic Kondo model.

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1. Introduction

Two dimensional quantum field theories with impurities or defects have received a great deal of attention in the past few years. In the $(1+1)$ quantum context, such theories have many important applications, including 'weak links' in infinite $s = 1/2$ Heisenberg Quantum Spin chains [1], local impurity potentials in interacting 1D electron systems (Quantum Wires)[2] [3], and tunneling point contacts in Fractional Quantum Hall Devices[4]. Notably, for the latter system, the methods of Exact Integrability as applied to impurities and defects have recently proven to be a powerful tool for providing non-perturbative answers to important strongly interacting quantum systems, observed in the Solid State laboratory[5]. In the 2D statistical mechanics context, the simplest example is the 2D Ising model in the full plane, with a defect interaction on the real axis [6][7][8].

Generally, a quantum field theory with defect can be formulated in terms of an action:

$$S_{\text{defect}} = \int_{-\infty<x<\infty} dxdt\ \mathcal{L}_{\text{bulk}} + \lambda \int dt\ D(0,t), \tag{1.1}$$

where $D(0,t)$ is a field operator located at the defect at $x = 0$. The integrability of such theories poses some special problems in comparison with boundary theories on the half-line $x > 0$[9]. If D is a local operator, then the action (1.1) breaks translation invariance; one thus expects the theory to have non-trivial transmission *and* reflection at the defect. Unfortunately, the algebraic Yang-Baxter like constraints involving both transmission and reflection have very limited solutions, and generally require the bulk theory to be a free field theory[10][8]. In particular, if D defines an integrable perturbation of the *bulk* CFT, this does not at all imply that the corresponding defect is integrable as well. This no-go theorem was circumvented in the works [3][4] by exploiting the following two special features of the bulk theory[2]. Namely, if the bulk theory is a massless conformal field theory (CFT), the defect theory can be folded onto a boundary field theory with twice the central charge (the tensor product of two identical copies of the original CFT), on the half-line. Secondly, when the bulk theory consists of a free scalar field, then the folded boundary theory consists of an even and odd combination of the original scalar field and the odd combination decouples from the boundary. (Thus, in this case, only half the central charge of the tensor product couples to the defect after folding, giving a boundary sine-Gordon model.) A larger class of theories that can be treated this way was

[2] consisting there of free massless scalar fields

studied in [11], and can be thought of as corresponding to defect operators that are purely chiral (left or right-moving); in this situation the theory is purely transmitting in the defect formulation, and the transmission S-matrices can be mapped onto the reflection S-matrices of the boundary formulation. (In some cases, this map requires the introduction of defect degrees of freedom.)

In this paper we consider a general situation where the bulk theory is a CFT, and the defect operator D is local (having both, left and right moving factors). By folding the system, we show that the class of integrable defect theories of this type is in one-to-one correspondence with integrable bulk perturbations of *two* copies of the CFT. A large class of such integrable bulk perturbations was identified in [12]. The resulting integrable theories include defects in minimal models and in coset theories[3] based on $SO(2n)$. As opposed to a defect in a theory of free scalar fields (as in [3],[4]), the *full* central charge of the tensor product of the original CFT couples to the defect after folding in this general situation. In this paper we focus in more detail on the case when the bulk is a $c < 1$ minimal unitary CFT and the defect operator is the primary field '$\Phi_{1,2}$' or '$\Phi_{2,1}$'. In the Ising case this corresponds to taking the spin field or the energy operator, respectively, as the defect perturbation D. For the spin perturbation, this corresponds to a line of magnetic field in the bulk of the sample. The case of the energy perturbation corresponds to a free field theory[4]. In contrast, as described below, the spin perturbation cannot be solved in the free fermion basis. Rather, it is related to the sine-Gordon theory at $\beta^2/8\pi = 1/8$, and a version of the anisotropic Kondo model, where these two cases correspond, as explained below in more detail, to different choices of boundary conditions in the ultra-violet ('continuous Neumann' and 'continuous Dirichlet'[13]).

We end this introduction by discussing a general conceptual aspect of integrable massless renormalization group (RG) flows of defect theories and of their corresponding boundary theories, obtained after folding. In general, a massless flow between two conformally invariant boundary conditions on a given CFT in the bulk is characterized by the following data: (i) the bulk CFT, (ii) the particular conformally invariant boundary condition chosen on this CFT before perturbation (i.e. the ultra-violet limit of the flow), and (iii) a particular relevant boundary operator chosen to perturb this boundary condition. Since a given

[3] These range from a $c = 1$ orbifold, to the level-one current algebra with $c = n$.

[4] For a massless bulk, this perturbation is exactly marginal and has been studied by many authors (see e.g. [6], [7], [13]), and the case of a massive bulk was considered in [8].

bulk CFT may in general have a large number of conformally invariant boundary conditions, several flows may be possible[5] (of course, always consistent with the 'g-theorem'[14]). If two or more of these flows are integrable, then, there must exist different reflection matrices, satisfying the bulk-boundary Yang-Baxter equations with the *same* bulk S-matrix, corresponding to all the possible integrable boundary flows. A complete classification of all these solutions is an open problem. However, the case of a spin-field defect in an Ising model, analyzed in section 4, is precisely an example of this non-trivial situation (perhaps the first to be understood completely). In this case we identify two integrable flows, connecting boundary conditions with different ratios g_{UV}/g_{IR} of 'ground state degeneracies' in the ultraviolet and the infrared . We find two different reflection matrices, one related to that of the boundary sine-Gordon model, the other related to the anisotropic Kondo model.

The paper is organized as follows: in Section 2 we briefly review basic ideas of integrability in the bulk and at the boundary, and discuss the folding procedure as applied to general defect theories in CFT's. Then we establish that the class of integrable defect theories is in one-to-one correspondence with integrable bulk perturbations of the tensor product of two copies a CFT. In section 3 we apply the general results of section 2 to the special case of defect perturbations of (Virasoro) minimal models. In section 4, we work out in detail the Ising case (the lowest minimal model). In particular, we obtain the bulk S-matrices, as well as the boundary reflection matrices for two different integrable massless boundary flows, and verify that those satisfy the bulk-boundary bootstrap, and give the correct values of the boundary entropies.

2. General Aspects of Integrability

In this section we outline a general strategy for constructing integrable local defect theories. We first review some features about bulk and boundary integrability that we will need.

[5] In the boundary sine-Gordon model there is only a single boundary flow, connecting the only possible conformally inv. boundary conditions on a free scalar field, von Neumann and Dirichlet.

2.1. Bulk and Boundary Integrability

One can define integrable bulk theories as suitable perturbations of conformal field theory (CFT)[15]:

$$S_{\rm bulk} = S^{\rm CFT}_{\rm bulk} + \Lambda \int dx dt \ \mathcal{O}(x,t). \tag{2.1}$$

Here, $S^{\rm CFT}_{\rm bulk}$ denotes a formal action for a specific CFT in the bulk, and $\mathcal{O}$ is a suitably chosen local perturbation making (2.1) integrable. For many infinite classes of CFT, the integrable perturbations $\mathcal{O}$ are known. For example, for the $c < 1$ minimal models, $\mathcal{O}$ can be the primary field $\Phi_{1,3}, \Phi_{1,2}$ or $\Phi_{2,1}$.

As usual, in Euclidean space let $z = t + ix$, $\overline{z} = t - ix$. The integrability of (2.1) implies there are an infinite number of conserved currents $J_L(z), J_R(\overline{z})$, which are chiral in the conformal limit and in the perturbed theory satisfy

$$\partial_{\overline{z}} J_L = \partial_z H, \qquad \partial_z J_R = \partial_{\overline{z}} \overline{H}, \tag{2.2}$$

for some $H, \overline{H}$. The local operator $\mathcal{O}(x,t)$ can be factorized into left and right moving components in the CFT:

$$\mathcal{O}(z,\overline{z}) = \mathcal{O}_L(z)\mathcal{O}_R(\overline{z}). \tag{2.3}$$

The conservation laws (2.2) are a consequence of the fact that for each $J_L(z)$ the residue of the operator product expansion of $J_L(z)$ with $\mathcal{O}_L(w)$ is a total derivative, and similarly for J_R and $\mathcal{O}_R$[15].

We now turn to boundary theories on the half line $x \geq 0$, with a boundary interaction at $x = 0$. In the conformal limit, the CFT must come equipped with a conformally invariant boundary condition satisfying $T_L(z) = T_R(\overline{z})$ at $x = 0$, where T_L, T_R are the left and right moving energy momentum tensors. This implies that at $x = 0$, the left and right moving operators are identified.

We make the following claim: *If the bulk theory (2.1) is integrable, then the boundary theory defined by*:

$$S_{\rm boundary} = S^{\rm CFT}_{\rm bound} + \lambda \int dt \ \mathcal{O}^{(L)}(0,t) \tag{2.4}$$

is also integrable.[6] (Here, $S^{\rm CFT}_{\rm bound}$ denotes the formal action of the bulk CFT including the conformally inv. boundary condition.) The reason is that the properties of the operator

[6] We distinguish between $\mathcal{O}_L$ on the infinite plane and its boundary counterpart $\mathcal{O}^{(L)}$.

product expansion of $J_L(z)$ with $\mathcal{O}_L$ described above, which ensure integrability of the bulk theory, also ensure that in the boundary theory one has

$$J^{(L)}(0,t) - J^{(R)}(0,t) = -i\partial_t \Theta, \tag{2.5}$$

for some Θ, and this implies that a conserved charge Q can be constructed from $J^{(L)}$, $J^{(R)}$:

$$Q = \int_0^\infty dx \left(J^{(L)} + J^{(R)} \right) + \Theta. \tag{2.6}$$

Using $\partial_{\overline{z}} J^{(L)} = \partial_z J^{(R)} = 0$, one easily sees that $\partial_t Q = 0$. One can prove (2.5) using conformal perturbation theory techniques outlined in[9].

2.2. Defect Theories

We now describe how to use the above facts to construct integrable local defect theories. Consider a defect theory on the full line $-\infty \leq x \leq \infty$ with a defect at $x = 0$. This can be formulated as a perturbation of a CFT:

$$S_{\text{defect}} = S^{\text{CFT}}_{\text{defect}} + \lambda \int dt \; D(0,t) \tag{2.7}$$

Here $S^{\text{CFT}}_{\text{defect}}$ denotes a bulk CFT equipped with a conformally inv. b.c. at the location of the defect, and $D(0,t)$ is an allowed operator at this defect. In the simplest case, discussed below, $S^{\text{CFT}}_{\text{defect}} = S^{\text{CFT}}_{\text{bulk}}$ is a bulk CFT without any defect at all[7]. In this case the perturbing field $D(0,t)$ is an operator of the bulk theory, placed at the location of the defect. We now fold the defect theory onto a boundary theory on the half line $x \geq 0$. In general, any bulk field Ψ can be decomposed into its components $\Psi^{(\pm)}$ on either side of the defect:

$$\Psi(x,t) = \Psi^{(+)}(x,t)\theta(x) + \Psi^{(-)}(x,t)\theta(-x). \tag{2.8}$$

Let $\psi_L(z)$ and $\psi_R(\overline{z})$ denote any left and right-moving fields, respectively, in the defect CFT, and $\psi_L^{(\pm)}(z)$, $\psi_R^{(\pm)}(\overline{z})$ their components for each side of the defect. From these we define four boundary fields in the region $x \geq 0$:

$$\begin{aligned} \psi_1^{(L)}(x,t) &= \psi_L^{(+)}(x,t), \qquad & \psi_1^{(R)}(x,t) &= \psi_L^{(-)}(-x,t) \\ \psi_2^{(L)}(x,t) &= \psi_R^{(-)}(-x,t), \qquad & \psi_2^{(R)}(x,t) &= \psi_R^{(+)}(x,t). \end{aligned} \tag{2.9}$$

[7] named "periodic b.c. " in [3]

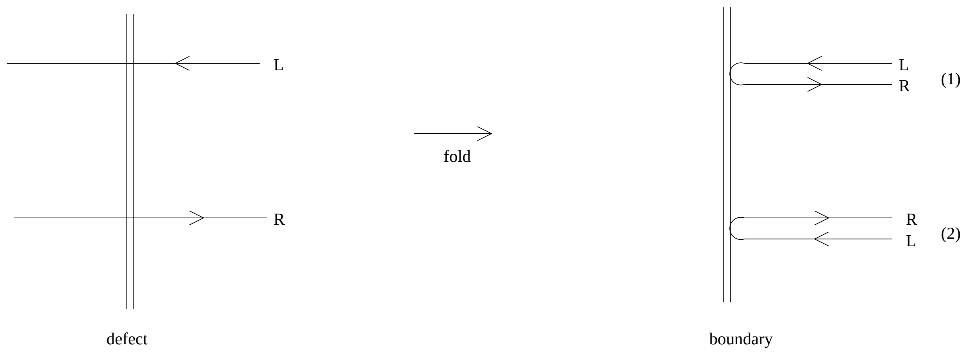

Figure 1. Graphical Representation of Folding

The boundary fields $\psi_{1,2}^{(L)}$ are functions of $z = t + ix$, whereas $\psi_{1,2}^{(R)}$ are functions of $\overline{z}$. This folding is represented graphically in figure 1.

For the energy momentum tensor, the defect conformal boundary condition is

$$T_L^{(-)}(0,t) = T_L^{(+)}(0,t), \qquad T_R^{(-)}(0,t) = T_R^{(+)}(0,t). \tag{2.10}$$

These imply the conformal boundary conditions

$$T_1^{(L)}(z) = T_1^{(R)}(\overline{z}), \qquad T_2^{(L)}(z) = T_2^{(R)}(\overline{z}), \qquad (x=0), \tag{2.11}$$

individually, on the two copies.

In the defect CFT we can factorize the perturbing field

$$D = D_L D_R. \tag{2.12}$$

To fold the theory, we let $D = (D^{(+)} + D^{(-)})/2$, with $D^{(\pm)} = D_L^{(\pm)} D_R^{(\pm)}$, and use the map (2.9). One obtains:

$$S_{\text{boundary}} = S_{\text{bound}}^{\text{CFT}_1 \otimes \text{CFT}_2} + \frac{\lambda}{2} \int dt \, \left(D_1^{(L)} D_2^{(R)} + D_1^{(R)} D_2^{(L)} \right), \tag{2.13}$$

where $S_{\text{bound}}^{\text{CFT}_1}$, $S_{\text{bound}}^{\text{CFT}_2}$ denote the two copies of the original CFT, with boundary conditions (2.11). We further assume that the boundary condition (2.11),(2.9),(2.8) allows us to

identify $D_1^{(L)} = D_1^{(R)}$, $D_2^{(L)} = D_2^{(R)}$ on the boundary; this is expected to be true up to some possible co-cycles. One then obtains

$$S_{\text{boundary}} = S_{\text{bound}}^{\text{CFT}_1 \otimes \text{CFT}_2} + \lambda \int dt\ D_1^{(L)} D_2^{(L)}, \tag{2.14}$$

The boundary thus couples the two copies of the CFT.

More generally, one may consider the unperturbed theory ($\lambda = 0$) in (2.14) to be equipped with *any* conformally inv. b.c. on the tensor product of the two copies of the CFT. Tracing back the steps, this defines the more general case[8] of a conformally inv. defect theory $S_{\text{defect}}^{\text{CFT}}$ in (2.7).

Based on the discussion in section 2.1, we can make the following statement. Let $\mathcal{C}$ denote the (bulk) CFT of the defect theory in (2.7). *Then the defect theory* (2.7) *is integrable if the following bulk perturbation of two copies of* $\mathcal{C}$ *is integrable*:

$$S_{\text{bulk}} = S_{\text{bulk}}^{\mathcal{C} \otimes \mathcal{C}} + \Lambda \int dxdt\ D_1 D_2, \tag{2.15}$$

where $D_{1,2}$ *are the local fields* D *from copies* $1, 2$ *of* $\mathcal{C}$.

In the situation when D is purely chiral: $D = D_L$, or $D_R = 1$, copy 2 of $\mathcal{C}$ decouples from the boundary (2.14). Thus, in this situation if $D_L D_R$ defines an integrable perturbation of *one* copy of $\mathcal{C}$, the defect theory is integrable. This is the situation studied in [16], and implicit in the works [3][4].

In the more interesting situation where D is local as in (2.12), the above requirements are extremely restrictive since they require known integrable perturbations of two copies of a CFT. Note in particular that if D defines an integrable *bulk* perturbation of one copy of $\mathcal{C}$, then this does not at all ensure that the defect version is also integrable. Nevertheless, there are large classes of such integrable perturbations. A first example was provided by Vaysburd who showed that two coupled minimal models are integrable[17]. A more general, and systematic scheme for coupling two (or more) copies of a conformal field theory in a way that leads to an integrable theory was described in [12], and is based on cutting and pasting of Dynkin diagrams for the associated affine Toda theories. This procedure leads to a large number of new and highly non-trivial massless integrable flows in defect theories.

We finish this section by describing two examples which are limiting cases of the integrable defect perturbations of all the minimal models considered in the next sections.

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[8] The perturbation of the 'continuous Neumann' boundary condition in the Ising case discussed below, is an exampe of this situation.

Ising model in a defect magnetic field. This model is defined by the action

$$S_{\text{defect}} = S^{\text{Ising}}_{\text{defect}} + \lambda \int dt \; \sigma(0,t), \tag{2.16}$$

where σ is the local spin field of dimension 1/8. Two copies of Ising is a $c = 1$ orbifold at the radius $R = 1$. An integrable perturbation of a scalar field is the sine-Gordon theory. Thus we expect the boundary version of this theory to be related to the boundary sine-Gordon theory at $\beta^2/8\pi = 1/8$ since it is at this coupling that the boundary perturbation has dimension 1/8. We will consider this theory in detail below.

$SU(2)$ Current Algebra at level 1. This model is defined by

$$S_{\text{defect}} = S_{k=1} + \lambda \int dt \sum_{m=\pm 1/2} \overline{\psi}^{(L)}_m \psi^{(R)}_m, \tag{2.17}$$

where $S_{k=1}$ is the $SU(2)$ WZW model at level 1 and $\psi^{(L)}_m$ is the primary field in the spinor representation with scaling dimension 1/4. Here $c = 1$. This current algebra can be bosonized, thus the integrability follows from the usual folding of free bosonic fields. The folded $c = 2$ theory is the $SO(4)$ level-one current algebra, and can thus be formulated as 4 real free fermions. Since the dimension of the perturbation is 1/2, the boundary version of this model is related to the boundary sine-Gordon theory at the free fermion point.

3. Defect Perturbations of Minimal Models

3.1. The Models

We now apply the ideas of the last section to defect perturbations of the $c < 1$ minimal series of unitary CFT. We let $\mathcal{C}_k$ denote the k-th minimal model with

$$c_k = 1 - \frac{6}{(k+2)(k+3)}, \tag{3.1}$$

$k = 1, 2, \ldots$ In $\mathcal{C}_k$ there exists the local primary fields $\sigma \equiv \Phi_{1,2}$ and $\widetilde{\sigma} \equiv \Phi_{2,1}$, with the scaling dimensions

$$\begin{aligned} \dim(\sigma) &= 2\Delta_\sigma = 2 \cdot \frac{1}{4}\left(1 - \frac{3}{k+3}\right) \\ \dim(\widetilde{\sigma}) &= 2\Delta_{\widetilde{\sigma}} = 2 \cdot \frac{1}{4}\left(1 + \frac{3}{k+2}\right). \end{aligned} \tag{3.2}$$

(Here, dim refers to the sum of the left and right conformal dimensions.) We define two defect theories, denoted $\mathcal{D}^{\sigma}_k$ and $\mathcal{D}^{\widetilde{\sigma}}_k$ which are defect perturbations of the minimal models by the above operators:

$$S_{\text{defect}} = S^{\mathcal{C}_k}_{\text{defect}} + \lambda \int dt \; \sigma(0,t), \tag{3.3}$$

and similarly with $\sigma \to \widetilde{\sigma}$.

3.2. Integrability

Upon folding, the defect theories $\mathcal{D}_k^\sigma$ become boundary theories, which we will denote $\mathcal{B}_k^\sigma$, with the action

$$S_{\text{boundary}} = S_{\text{bound}}^{\mathcal{C}_k \otimes \mathcal{C}_k} + \lambda \int dt\ \sigma_1^{(L)} \sigma_2^{(L)}. \tag{3.4}$$

All of the statements of this section apply with $\sigma \to \widetilde{\sigma}$, and are implied. The arguments of the previous section indicate that $\mathcal{D}_k^\sigma$ are integrable if the bulk theories which are defined as bulk perturbations of $\mathcal{C}_k \otimes \mathcal{C}_k$ by $\mathcal{O} = \sigma_1 \sigma_2$ are integrable. We will refer to these bulk theories as $\mathcal{M}_k^\sigma$.

Remarkably, the bulk theories $\mathcal{M}_k^\sigma$ are in fact integrable[17][12]. One way to explain this is as follows. The $\mathcal{C}_k$ minimal model can be formulated as an $SU(2)$ coset:

$$\mathcal{C}_k = \frac{SU(2)_k \otimes SU(2)_1}{SU(2)_{k+1}}, \tag{3.5}$$

where $SU(2)_k$ is the WZW model at level k. Now we use the fact that

$$SU(2)_k \otimes SU(2)_k = SO(4)_k. \tag{3.6}$$

This implies

$$\mathcal{C}_k \otimes \mathcal{C}_k = \frac{SO(4)_k \otimes SO(4)_1}{SO(4)_{k+1}}. \tag{3.7}$$

As explained by Vaysburd, there is an unconventional way in which to affinize $SO(4)$, extending the Dynkin diagram by the highest weight of the vector representation rather than adjoint, leading to the twisted affine algebra $d_3^{(2)}$.[9] The spectrum and S-matrices of the bulk theories $\mathcal{M}_k^\sigma$ can be obtained as RSOS restrictions of the dual $c_2^{(1)}$ affine Toda theory which has quantum affine symmetry ${}_q d_3^{(2)}$, as was done in [12].

The limiting cases are the Ising model in a defect magnetic field (2.16), which occurs at $k = 1$, and the current algebra with defect (2.17), occurring at $k = \infty$. It was shown in [17][12] that the bulk theory $\mathcal{M}_{k=\infty}^\sigma$ is equivalent to 4 real massive free fermions, and this is consistent with our previous remarks for the model (2.17).

To solve the folded defect theories $\mathcal{B}_k^\sigma$, one must start with the bulk spectrum of particles that diagonalizes the boundary interaction; this spectrum is dictated by the bulk theory $\mathcal{M}_k^\sigma$. Given this spectrum and the bulk massless S-matrices, one then finds boundary reflection S-matrices that are consistent with the algebraic constraints described

[9] One has the identification $a_3^{(2)} = d_3^{(2)}$.

in the next section. Alternatively one can think of solving the bulk massive theory $\mathcal{M}_k^\sigma$ with the boundary interaction of $\mathcal{B}_k^\sigma$, and then taking the bulk massless limit $\Lambda \to 0$, as was done for sine-Gordon in [18]; this is more complicated however since the boundary Yang-Baxter equation is more complicated in the massive versus massless case. The bulk S-matrices for the models $\mathcal{M}_k^\sigma$ are mostly known[17][12], and in general have an RSOS form. In the next section, we work out the Ising case where the bulk S-matrix is diagonal.

4. Ising Case

In this section we work out the Ising case at $k = 1$. The defect problem is described by the action (2.16). By the arguments of the last section, we must first consider the folded boundary theory with the action

$$S_{\rm bound} = S_{\rm bound}^{{\rm Ising}_1 \otimes {\rm Ising}_2} + \Lambda \int_{x \geq 0} dxdt \; \sigma_1 \sigma_2 + \frac{\lambda}{2} \int dt \; \sigma_1^{(L)} \sigma_2^{(L)}, \tag{4.1}$$

where the subscripts refer to copies $1, 2$ of the Ising CFT. As explained above, the original defect theory corresponds to massless bulk term $\Lambda = 0$, and the presence of Λ only serves to determine the bulk spectrum which diagonalizes the boundary interaction.

It is important to realize that a theory is not completely defined by the action (4.1); the theory is only completely specified once the boundary CFT in the ultra-violet is equipped with a conformal boundary condition. The Ising $\otimes$ Ising CFT is equivalent to a $c = 1$ orbifold at radius 1. The possible conformal boundary conditions for the orbifold are richer than for the non-orbifolded theory and were classified in [13]. There it was shown that the possible boundary conditions are 'continuous Neumann' and 'continuous Dirichlet' depending on the continuous parameters $\widetilde{\phi}_0$ and ϕ_0 respectively, as well as tensor products of the known free and fixed Ising boundary conditions. For a generic non-orbifolded scalar field ϕ, on the other hand, the only possible conformal boundary conditions are Neumann and Dirichlet and the dependence on zero modes $\phi_0, \widetilde{\phi}_0$ can be removed by a shift in ϕ; for the orbifold case this is not possible because of the identification $\phi \sim -\phi$.

The Ising case has additional complexity due to the existence of exactly marginal bulk and boundary directions which do not exist for higher k. Namely, consider adding to the action (4.1) the terms

$$\delta S = \Lambda' \int_{x \geq 0} dxdt \; \varepsilon_1 \varepsilon_2 + \lambda' \int dt \; \varepsilon_1^{(L)} \varepsilon_2^{(L)}, \tag{4.2}$$

where ε is the energy operator with scaling dimension 1. (In terms of Majorana fermions, $\varepsilon = \chi_L \chi_R$.) Both Λ' and λ' have scaling dimension zero, and are actually completely marginal. The parameter Λ' corresponds to moving along the Ashkin-Teller line of bulk fixed points, and corresponds to a modification of the sine-Gordon coupling β below. The parameter λ' can be shown[13] to correspond to the parameters ϕ_0 and $\widetilde{\phi}_0$ of the 'continuous Dirichlet' and the 'continuous Neumann' boundary conditions. The dimension of the defect operator (coupling λ, in (4.1)) varies continuously with these latter parameters. Henceforth we assume that Λ' and λ' are zero and consequently the dimension of the defect operator σ is 1/8. Equivalently, the parameters ϕ_0 and $\widetilde{\phi}_0$ are taken to be fixed to the values corresponding to $\lambda' = 0$.

A particular scattering theory describes a flow between two conformal boundary conditions. One expects that in the infrared when $\lambda \to \infty$ the two copies of Ising each have a fixed boundary condition, i.e. in the infra-red the boundary condition is fixed-fixed. Below we will propose two scattering theories which possess either the 'continuous Neumann' or the 'continuous Dirichlet' boundary condition in the ultra-violet. We first describe the bulk theory and the constraints on boundary massless scattering.

4.1. Bulk Theory

The bulk theory is a special case of the coupled minimal models studied in [17][12]. The bulk massive S-matrices are the same as for the sine-Gordon theory at $\beta^2/8\pi = 1/8$,[10] up to some minus signs in the soliton sector[12]. These signs can be traced to the fact that Ising $\otimes$ Ising is an orbifold CFT of a scalar field[19]. The sine-Gordon theory at this coupling is described by the bulk action

$$S = \frac{1}{4\pi} \int dxdt \left(\frac{1}{2} \left(\partial_\mu \phi\right)^2 + \Lambda \cos\left(\phi/2\right) \right). \tag{4.3}$$

The spectrum consists of two solitonic particles s_1, s_2, and 6 breathers with mass ratios

$$m_a = 2m_s \sin \frac{a\pi}{14}, \qquad a = 1, 2, .., 6. \tag{4.4}$$

In contrast to the sine-Gordon case where the solitons s_1, s_2 carry $U(1)$ charge ± 1, here the solitons are not charge conjugates of each other, but rather are their own anti-particle. The bulk S-matrices are

$$S_{s_1 s_1} = S_{s_2 s_2} = S_{s_1 s_2} = F_{\frac{1}{7}}(\theta) F_{\frac{2}{7}}(\theta) F_{\frac{3}{7}}(\theta), \tag{4.5}$$

[10] The coupling β is normalized in the conventional way where the free fermion point occurs at $\beta^2 = 4\pi$.

where

$$F_\alpha(\theta) = \frac{\tanh\frac{1}{2}(\theta + i\pi\alpha)}{\tanh\frac{1}{2}(\theta - i\pi\alpha)}, \tag{4.6}$$

and θ is the rapidity $E = m\cosh\theta, P = m\sinh\theta$. For the breathers one has

$$\begin{aligned} S_{ab}(\theta) &= \left(\frac{|a-b|}{14}\right)\left[\prod_{k=1}^{\min(a,b)-1}\left(\frac{|a-b|+2k}{14}\right)\right]^2\left(\frac{a+b}{14}\right) \\ S_{as_1}(\theta) &= S_{as_2}(\theta) = (-1)^a\prod_{k=0}^{a-1}\left(\frac{7-a+2k}{14}\right), \end{aligned} \tag{4.7}$$

where $(\alpha) \equiv F_\alpha(\theta)$.

The structure of bound states also differs from the sine-Gordon theory; in [12] this structure was obtained from the restricted ${}_q d_3^{(2)}$ symmetry. For our problem only the even breathers $(2,4,6)$ are $s_1 - s_1$ or $s_2 - s_2$ bound states. This is to be compared with the sine-Gordon case where all breathers are $s_1 - s_2$ bound states.

4.2. Algebraic constraints on massless boundary scattering

We will need the the algebraic constraints on the massless boundary scattering matrices. These can be obtained by an appropriate massless limit of the equations in [9], as we now describe.

In the massive case the boundary reflection S-matrices R_a^b satisfy the crossing and unitarity constraints:

$$\begin{aligned} R_a^c(\theta)R_c^b(-\theta) &= \delta_a^b \\ K^{ab}(\theta) &= S_{a'b'}^{ab}(2\theta)K^{b'a'}(-\theta), \end{aligned} \tag{4.8}$$

where $K^{ab}(\theta) = R_{\bar{a}}^b(i\pi/2 - \theta)$. The massless limit can be taken by replacing $\theta \to \theta + \alpha$, and letting $\alpha \to \infty$ and $m \to 0$ keeping $me^\alpha/2 = \mu$ held fixed. This gives the dispersion relation for right movers $E_a = \mu_a e^\theta$, $P_a = \mu_a e^\theta$, where the μ_a have the same ratios as the bulk masses(4.4). To obtain left-movers, one lets $\theta \to \theta - \alpha$, and takes the same limits leading to $E_a = \mu_a e^{-\theta}$, $P_a = -\mu_a e^{-\theta}$. Thus, given some reflection S-matrices R_a^b in a massive theory, we define the massless scattering matrices as follows:

$$\begin{aligned} \widetilde{\mathcal{R}}_a^b(\theta) &= \lim_{\alpha\to\infty, m\to 0} R_a^b(\theta + \alpha), \qquad \text{for right movers} \\ \mathcal{R}_a^b(\theta) &= \lim_{\alpha\to\infty, m\to 0} R_a^b(\theta - \alpha), \qquad \text{for left movers.} \end{aligned} \tag{4.9}$$

The right hand sides of (4.9) depend on $\theta-\theta_B$ for right-movers, and $\theta+\theta_B$ for left-movers, where μe^{θ_B} is defined as a physical boundary energy scale, as described in [18]. The bulk S-matrix in the massless limit becomes an S-matrix S_{LL} (S_{RR}) for left (right) movers, both of which are the same as (4.7). We will also need the 'braiding' matrix:

$$B_{ab}^{cd}=\lim_{\theta\to-\infty}S_{ab}^{cd}(\theta). \tag{4.10}$$

In general B is a solution of the braiding relations.

The two equations (4.8) become

$$\begin{aligned}&\mathcal{R}_a^c(\theta)\widetilde{\mathcal{R}}_c^b(-\theta)=\delta_a^b\\&\widetilde{\mathcal{R}}_{\overline{a}}^{b}\left(i\pi/2-\theta\right)=B_{cd}^{ab}\;\mathcal{R}_{\overline{d}}^{c}\left(i\pi/2+\theta\right).\end{aligned} \tag{4.11}$$

These can be combined into a single equation for $\mathcal{R}$:

$$B_{cd}^{eb}\;\mathcal{R}_a^{\overline{e}}\left(\theta-i\pi/2\right)\mathcal{R}_{\overline{d}}^{c}\left(\theta+i\pi/2\right)=\delta_a^b. \tag{4.12}$$

For diagonal bulk scattering,

$$B_{cd}^{ab}=B_{ab}\delta_c^a\delta_d^b, \tag{4.13}$$

one has

$$\sum_c B_{cb}\;\mathcal{R}_a^{\overline{c}}\left(\theta-i\pi/2\right)\mathcal{R}_{\overline{b}}^{c}\left(\theta+i\pi/2\right)=\delta_a^b. \tag{4.14}$$

Another constraint comes from the boundary bootstrap. If the particle of type c is a bound state of particles a,b then the bulk S-matrix has a pole at iu_{ab}^c:

$$S_{ab}^{a'b'}(\theta)\approx i\frac{f_{ab}^c f_c^{a'b'}}{\theta-iu_{ab}^c}. \tag{4.15}$$

The fusing angles satisfy $u_{ab}^c+u_{bc}^a+u_{ac}^b=2\pi$. Taking the massless limit of the boundary bootstrap equation in [9] one obtains

$$f_c^{ab}\mathcal{R}_d^c(\theta)=f_d^{b_1a_1}\mathcal{R}_{a_1}^{a_2}(\theta+i\overline{u}_{ac}^b)\mathcal{R}_{b_2}^b(\theta-i\overline{u}_{bc}^a)B_{b_1a_2}^{b_2a}, \tag{4.16}$$

where $\overline{u}=\pi-u$. For diagonal bulk scattering,

$$f_c^{ab}\mathcal{R}_d^c(\theta)=f_d^{b'a'}\mathcal{R}_{a'}^{a}(\theta+i\overline{u}_{ac}^b)\mathcal{R}_{b'}^b(\theta-i\overline{u}_{bc}^a)B_{b'a}. \tag{4.17}$$

A subset of the fusing angles we used to check our solutions below are the following:

$$
\begin{aligned}
u^2_{s_1 s_1} &= \frac{5\pi}{7}, \qquad u^4_{s_1 s_1} = \frac{3\pi}{7}, \qquad u^6_{s_1 s_1} = \frac{\pi}{7} \\
u^2_{11} &= \frac{\pi}{7}, \qquad u^4_{22} = \frac{2\pi}{7}, \qquad u^3_{12} = \frac{3\pi}{14}.
\end{aligned}
\tag{4.18}
$$

4.3. Boundary reflection S-matrices

In our problem the bulk theory has the properties:

$$
B_{s_1 s_1} = B_{s_2 s_2} = B_{s_1 s_2} = 1, \qquad \overline{s_1} = s_1, \ \overline{s_2} = s_2, \tag{4.19}
$$

whereas for sine-Gordon $B_{s_i s_j} = -1$ and $\overline{s_1} = s_2$. We now describe two boundary scattering theories that are both consistent with the above constraints.

Boundary Sine-Gordon-like Solution In [18] reflection S-matrices for the boundary sine-Gordon (BSG) theory were obtained as the massless limit of the results in [9][20]. We find that by modifying some phases in these reflection S-matrices we can continue to satisfy (4.14) with the new conditions (4.19), and also the bootstrap equation(4.17). The result is

$$
\begin{aligned}
\mathcal{R}^{s_2}_{s_1} = \mathcal{R}^{s_1}_{s_2} &= \frac{e^{-7\theta/2}}{2\cosh(7\theta/2 - i\pi/4)} \ e^{i\delta'} Y(\theta) \\
\mathcal{R}^{s_1}_{s_1} = \mathcal{R}^{s_2}_{s_2} &= \frac{e^{7\theta/2}}{2\cosh(7\theta/2 - i\pi/4)} \ e^{i\delta} Y(\theta) \\
\mathcal{R}_{2k}(\theta) &= (-1)^{k-1} \prod_{l=1}^{k} F_{\frac{2l-8}{14}}(\theta) \\
\mathcal{R}_{2k-1}(\theta) &= i(-1)^{k-1} f_{-\frac{1}{2}}(\theta) \prod_{l=1}^{k-1} F_{\frac{2l-7}{14}}(\theta),
\end{aligned}
\tag{4.20}
$$

where for the breathers $\mathcal{R}_a \equiv \mathcal{R}^a_a$, and

$$
\begin{aligned}
Y(\theta) &= F_{-\frac{3}{14}}(\theta) f_{-\frac{1}{2}}(\theta), \qquad e^{i\delta} = e^{-i\delta'} = e^{i\pi/4} \\
f_\alpha(\theta) &\equiv \frac{\sinh\frac{1}{2}(\theta + i\pi\alpha)}{\sinh\frac{1}{2}(\theta - i\pi\alpha)}.
\end{aligned}
\tag{4.21}
$$

(For the boundary sine-Gordon theory one has instead $e^{i\delta'} = i, \ e^{i\delta} = 1$.) The constraints of crossing-unitarity(4.14) and the bootstrap(4.17), are easily checked using $F_\alpha = -f_\alpha f_{1-\alpha} = F_{1-\alpha}$, $f_\alpha = f_{\alpha+2}$, $f_\alpha f_{-\alpha} = 1$, and $f_\alpha(\theta + i\pi\beta) f_\alpha(\theta - i\pi\beta) = f_{\alpha-\beta} f_{\alpha+\beta}$.

Kondo-like solution Another solution starts from the minimal one in the soliton sector:

$$\mathcal{R}_{s_1}^{s_1} = \mathcal{R}_{s_2}^{s_2} = i f_{-\frac{1}{2}}(\theta) = \tanh \frac{1}{2}(\theta - i\pi/2), \qquad \mathcal{R}_{s_1}^{s_2} = \mathcal{R}_{s_2}^{s_1} = 0. \tag{4.22}$$

Closing the boundary bootstrap on the breathers using (4.17) gives

$$\mathcal{R}_a(\theta) = F_{-a/14}(\theta). \tag{4.23}$$

This is essentially the same as for the anisotropic Kondo model[21][22], except that in the latter $R_{s_1}^{s_1} = R_{s_2}^{s_2} = 0.$[11].

4.4. Boundary Entropy

In order to determine the ultraviolet (UV) and infra-red (IR) fixed points of the (massless) flows described by the above scattering theories, we examine the so-called 'ground state degeneracies' g[14].

Consider the partition function $Z_{\alpha\alpha'}$ on a cylinder of circumference L and length R, with boundary conditions α and α' at the ends of the cylinder. If one formulates this in a picture where the hamiltonian evolves the system in the direction along the length of the cylinder, then

$$Z_{\alpha\alpha'} = \langle B_\alpha | e^{-HR} | B_{\alpha'} \rangle, \tag{4.24}$$

where $|B_{\alpha,\alpha'}\rangle$ are boundary states. In the limit of large ratio R/L,

$$Z_{\alpha\alpha'} = g_\alpha g_{\alpha'} \to \langle B_\alpha | 0 \rangle \langle 0 | B_{\alpha'} \rangle, \tag{4.25}$$

where g_α is the 'ground state degeneracy' for the boundary condition α.

For the Ising$\otimes$Ising theory, it is known[13]that the 'continuous Neumann', 'continuous Dirichlet', and 'fixed-fixed' conformal boundary conditions have 'ground state degeneracies' $g = \sqrt{2}$, $g = 1$, and $g = 1/2$ respectively.

For the scattering theory the ratio of ultra-violet to infra-red boundary entropies can be computed using the boundary version of the thermodynamic Bethe ansatz (TBA) [18][23]. If the scattering theory is diagonal, the result is

$$\log \frac{g_{UV}}{g_{IR}} = \sum_a \frac{1}{2\pi i} \int d\theta \partial_\theta \left(\log \mathcal{R}_a\right) \log\left(1 + e^{-\varepsilon_a(\theta)}\right), \tag{4.26}$$

[11] We thank H. Saleur for suggesting this possibility.

where ε_a are the bulk TBA pseudo-energies for particle of type a, satisfying the integral equations in [24].

For our problem, the analysis proceeds much as in [18], where the flow in the boundary entropy was computed for the BSG theory at the reflectionless points. One finds

$$\log \frac{g_{UV}}{g_{IR}} = \sum_{a=1}^{6} I^{(a)} \log(1 + 1/x_a) + (I^{(+)} + I^{(-)}) \log(1 + 1/x_\pm), \tag{4.27}$$

where x_n are related to the constant ultra-violet values of ε_n, $x_n \equiv \exp(\varepsilon_n(\infty))$, and $I^{(n)} = \int d\theta \partial_\theta \mathcal{R}_n(\theta)/2\pi i$. The $I^{(\pm)}$ and $x_\pm$ come from the soliton sector; in the BSG case the boundary scattering is diagonal in the basis $(\pm) = s_1 \pm s_2$. The x_n are bulk properties and are known to be $x_a = (a+1)^2 - 1$ and $x_\pm = 7$.

It is not difficult to show that the effect of the phase differences in (4.20) in comparison to the BSG case is merely to distribute the contributions to g from the solitons differently, but does not modify the sum of the contributions from s_1 and s_2. Namely, $I^{(+)}+I^{(-)} = 7/2$ and $I^{(a)} = a/2$. Thus for the BSG-like solution(4.20) one has

$$\frac{g_{UV}}{g_{IR}} = \left(\frac{8\pi}{\beta^2}\right)^{1/2} = 2\sqrt{2}. \tag{4.28}$$

We thus conjecture that this describes the flow between the continuous Neumann and fixed-fixed boundary condition.

In the Kondo-like solution(4.22)(4.23), one finds instead $I^{(a)} = 1$, $I^{(s_1)} = I^{(s_2)} = 1/2$. This leads to

$$\frac{g_{UV}}{g_{IR}} = 2. \tag{4.29}$$

Here, the scattering theory is conjectured to describe the flow between a continuous Dirichlet and fixed-fixed boundary condition.

Further arguments in favor of these conjectures are as follows. Though our bulk theory is an orbifolding of the sine-Gordon theory which modes out the $U(1)$ symmetry by Z_2, this does not modify significantly the bulk S-matrices, and this suggests that the boundary reflection S-matrices also have $U(1)$ properties that are similar to the BSG case. It is known that for the non-orbifolded BSG theory the Neumann boundary condition breaks the $U(1)$ symmetry and this allows $\mathcal{R}_{s_1}^{s_2} \neq 0$, as in (4.20). On the other hand, the Dirichlet boundary condition preserves the $U(1)$ symmetry, as does (4.22).

We end with a discussion of the effect of the continuous parameter characterizing the 'continuous Dirichlet' fixed line. As one moves along this continuous line of ultraviolet fixed

points of the flow, parametrized by φ_0, the scaling dimension $\Delta_b(\varphi_0)$ of the perturbing boundary operator varies continuously with φ_0. This is reminiscent of the situation in the anisotropic Kondo model. Indeed, we propose that the boundary reflection $S-$matrices are those of the anisotropic Kondo model at the appropriate value of β, namely[12] $\beta^2/8\pi = \Delta_b(\varphi_0)$. Note that the anisotropic Kondo boundary $S-$matrix leads to a ratio of (ground state degeneracy) g-values $g_{UV}/g_{IR} = 2$, *independent* of the value of β, precisely as required for the flow from the 'continuous Dirichlet' fixed line to the fixed-fixed Ising boundary condition.

5. Conclusions

We have described a framework for constructing integrable massless quantum field theories with local defects which generalizes the folding technique for obtaining integrable defect theories, that was previously understood only for free scalar fields. The classification of such theories parallels the classification of integrable bulk perturbations of two copies of a conformal field theory, as pursued in [12], and corresponds to a new class of integrable theories, obtained by using an analysis of the (extended) Dynkin diagram of affine Lie algebras. This approach has allowed us to propose a solution to the problem of an Ising model in a defect magnetic field. The Ising case extends to all the minimal models of unitary conformal field theory with magnetic defects. The more general class includes also defects in all coset minimal models based on $SO(2n)$.

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[12] Equation (4.23)is to be replaced with $\mathcal{R}_a(\theta) = F_{-ag/(2-2g)}(\theta)$, where $g = \beta^2/8\pi$. Equation (4.22), on the other hand, remains unchanged.

References

[1] I Affleck and S Eggert, Phys. Rev. Lett. 75 (1995) 934.

[2] C.L. Kane and M.P.A. Fisher, Phys. Rev. B 46 (1992) 15233

[3] E. Wong and I. Affleck, Nucl. Phys. B417 (1994) 403.

[4] P. Fendley, A. W. W. Ludwig, and H. Saleur, Phys. Rev. Lett. 74 (1995) 3005; Phys. Rev. B52 (1995) 8934; Statphys. 19, p.137 (World Scientific, 1996).

[5] F.P. Milliken, C.P. Umbach and R.A. Webb, Solid State Commun. 97 (1996) 309.

[6] R. Bariev, Sov. Phys. JETP 50 (1979) 613.

[7] B. McCoy and J. H. H. Perk, Phys. Rev. Lett. 44 (1980) 840.

[8] G. Delfino, G. Mussardo and P. Simonetti, Nucl. Phys. B432 (1994) 518.

[9] S. Ghoshal and A. Zamolodchikov, Int. J. Mod. Phys. A9 (1994) 3841.

[10] P. Fendley, talk given at the conference on 'Statistical Mechanics and Quantum Field Theory', Univ. of Southern California, 16-2- May 1994; and private communication.

[11] R. Konik and A. LeClair, *Purely Transmitting Defect Field Theories*, hep-th/9703085.

[12] A. LeClair, A. Ludwig and G. Mussardo, *Integrability of Coupled Minimal Models*, ITP preprint ITP-97-081.

[13] M. Oshikawa and I. Affleck, Phys.Rev.Lett. 77 (1996) 2604; Nucl.Phys. B495 (1997) 533-582.

[14] I. Affleck and A. Ludwig, Phys. Rev. Lett. 67 (1991) 161.

[15] A. Zamolodchikov, Int. J. Mod. Phys. A3 (1988) 743.

[16] R. Konik and A. LeClair, *Purely Transmitting Defect Field Theories*, hep-th/9703085.

[17] I. Vaysburd, Nucl. Phys. B446 (1995) 387.

[18] P. Fendley, H. Saleur and N. Warner, Nucl. Phys. B430 (1994) 577.

[19] P. Ginsparg, Nucl. Phys. B 295 (1988) 153.

[20] S. Ghoshal, Int. J. Mod. Phys. A9 (1994) 4801.

[21] P. Fendley, Phys. Rev. Lett. 71 (1993) 2485.

[22] F. Lesage, H. Saleur and S. Skorik, Nucl. Phys. B474 (1996) 602.

[23] A. LeClair, G. Mussardo, H. Saleur and S. Skorik, Nucl. Phys. B453 (1995) 581.

[24] Al. Zamolodchikov, Nucl. Phys. B342 (1991) 695.